Super-quadratic Lower Bounds for Depth-2 Linear Threshold Circuits

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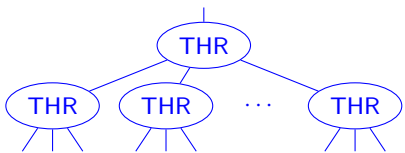
Apr 16, 2026

Today's Plan

- **1. Backgrounds in TC^0 circuit lower bounds**
- **2. Two main ideas behind our new lower bound:**
  - Random Restrictions in the Algorithmic Method
  - List-Approximation Polynomials

The Problem Setting - Threshold Circuits (TC^0)

TC^0: polynomial-size, **constant-depth** circuits with **Linear Threshold Gates** and free NOT gates



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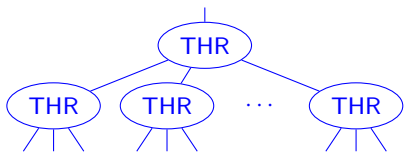

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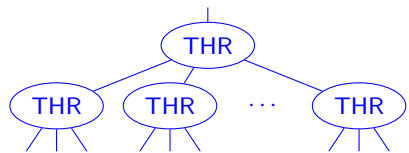

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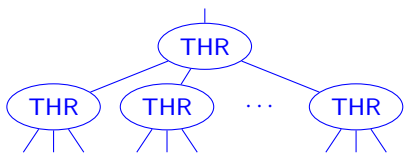

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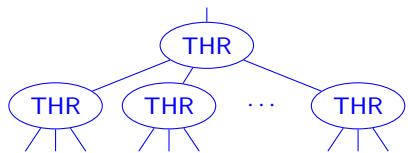

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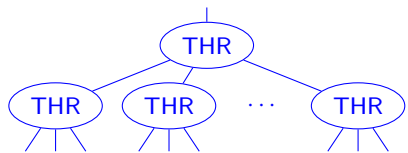

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Big Open Problem: $\mathsf{NEXP} \overset{?}{\not\subseteq} \mathsf{TC}^0$, or even $\mathsf{E}^{\mathsf{NP}} \overset{?}{\not\subseteq} \mathsf{TC}^0$ (Although we believe $\mathsf{P} \not\subseteq \mathsf{TC}^0$)

Depth-2 Threshold Circuit Lower Bounds

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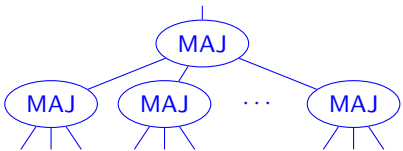	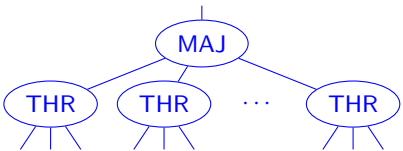
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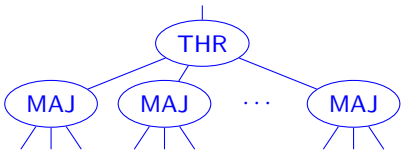	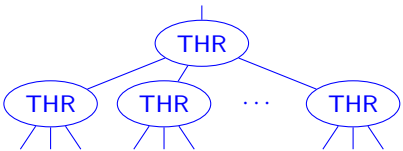

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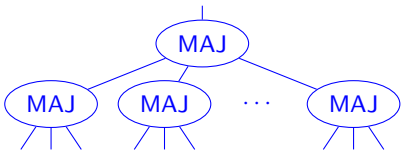	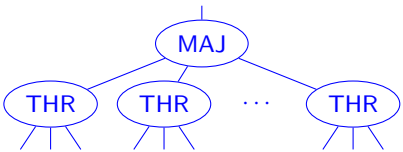
InnerProduct $\notin$ MAJ ∘ MAJ [HMPST' 93]	InnerProduct $\notin$ MAJ ∘ THR [Nisan' 94]
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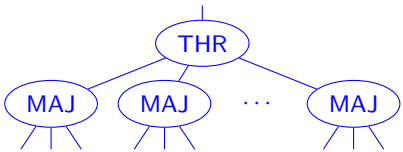	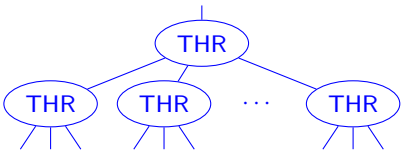
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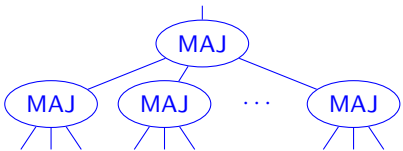
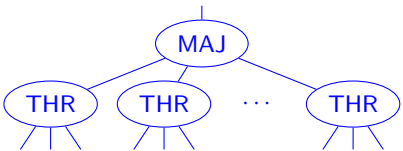
 InnerProduct $\notin$ MAJ ∘ MAJ [HMPST' 93]	 InnerProduct $\notin$ MAJ ∘ THR [Nisan' 94]
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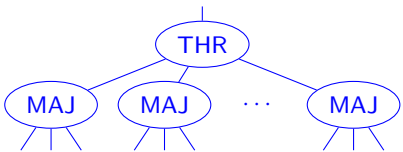
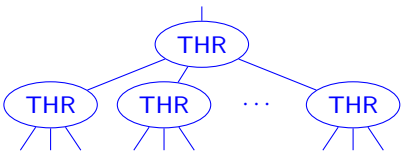
 InnerProduct $\notin$ THR ∘ MAJ [Forster' 02]	 Open: $\mathsf{E}^{\mathsf{NP}} \overset{?}{\not\subseteq}$ THR ∘ THR

Approaches Towards Circuit Lower Bounds

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- CAPP (Circuit Acceptance Probability Problem)
  - Input: A Circuit
  - Output: Estimate the fraction of inputs in $\{0,1\}^n$ that outputs 1
  - Error: $\pm o(1)$ (arbitrarily small constant)
- Non-trivial: runs in deterministic $2^n/n^{\omega(1)}$ time for n-input poly(n)-size circuits. (Trivial is $2^n \cdot \text{poly}(n)$.)

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Our Result

- Previous Results:

$\mathsf{P} \not\subseteq n^{1.5-\varepsilon}$-size-$\mathsf{THR} \circ \mathsf{THR}$ [Kane-Williams' 16]

$\mathsf{E}^{\mathsf{NP}} \not\subseteq n^{2-\varepsilon}$-size-$\mathsf{THR} \circ \mathsf{THR}$ [Alman-Chan-Williams' 16] [Tamaki' 16]

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via designing an algorithm for

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- **2. Two main ideas behind our new lower bound:**
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Idea 1: Random Restrictions in the Algorithmic Method

- A standard step in designing circuit analysis algorithms: partition the input into two parts:

$$x \in \{0,1\}^n \rightarrow \boxed{|y| = n^{\varepsilon/10}} \boxed{|z| = n - n^{\varepsilon/10}}$$

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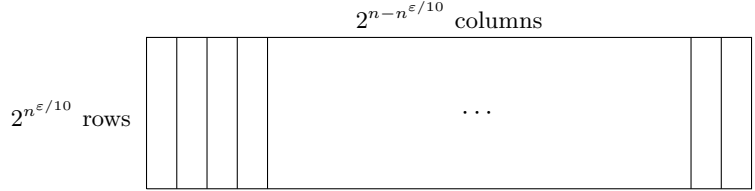

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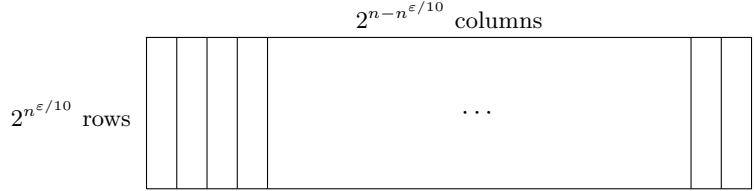

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- **Observation:** Each column can be viewed as a restriction!

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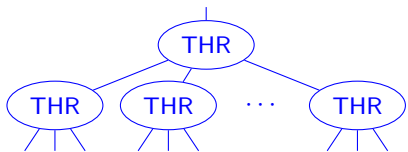

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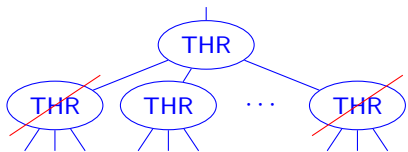

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- **Proof:** A Toy example: n-bit MAJ gate
  - If non-constant, then

$$|\#\text{fixed 1-s} - \#\text{fixed 0-s}| \leq k$$

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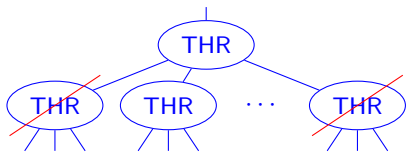

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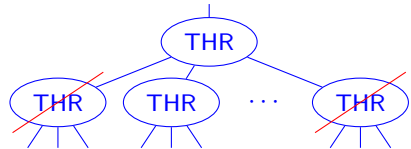

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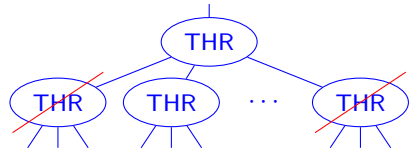

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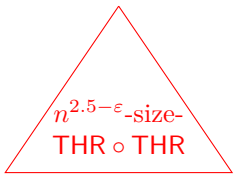

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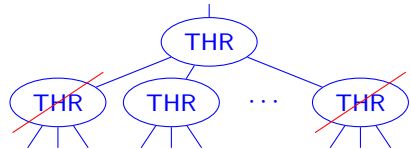

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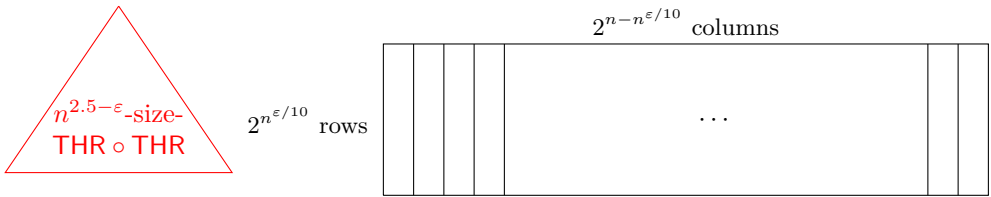

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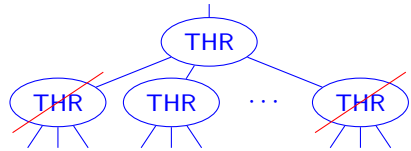

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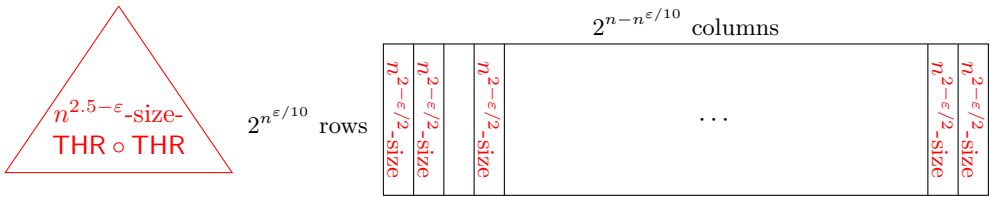

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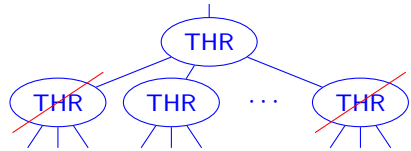
Simplification of Threshold Gates under Random Restrictions

- **Random Restriction:** randomly set $k := n^{\varepsilon/10}$ input bits to be $\star$ (free variables), set each remaining bit to be fixed 0 or 1 randomly.
- **Theorem:** A THR gate remains non-constant w.p. $\approx k/\sqrt{n}$.



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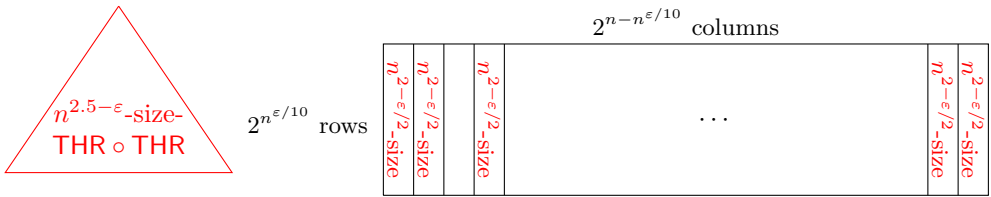

- Most columns become $n^{2-\varepsilon/2}$-size THR $\circ$ THR circuits!
- Issue: the $n^{2-\varepsilon/2}$-size circuits of each column are different.

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- Why $n^{2-\varepsilon}$ in previous results?

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- A CAPP algorithm for $\mathsf{POLY}[n^{1-\Omega(\varepsilon)}] \circ \mathsf{THR}$.

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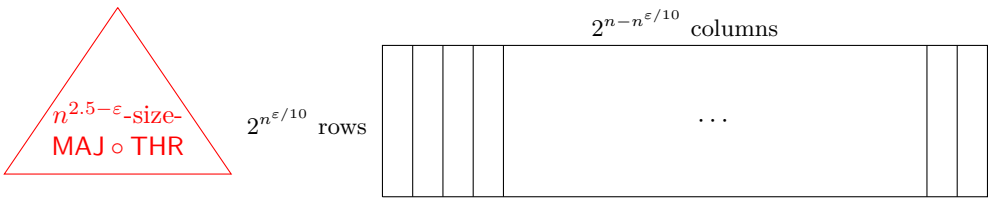

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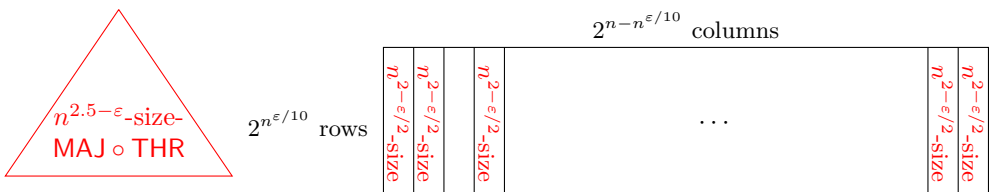

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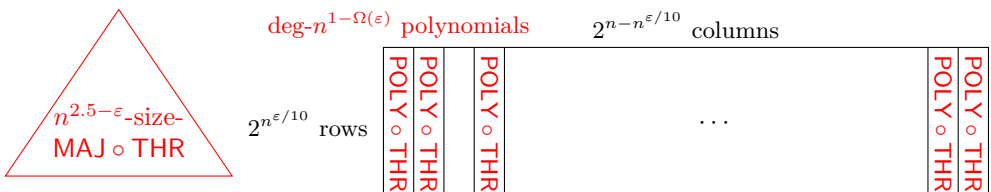

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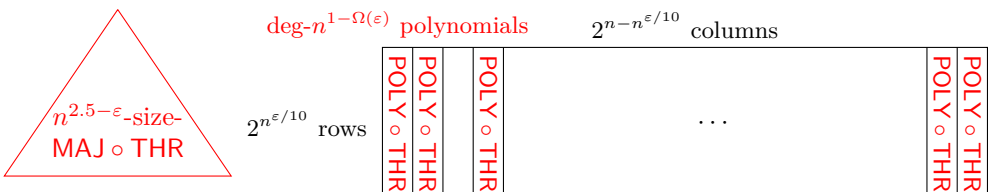

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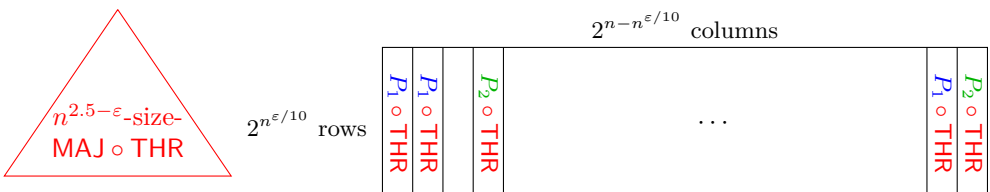

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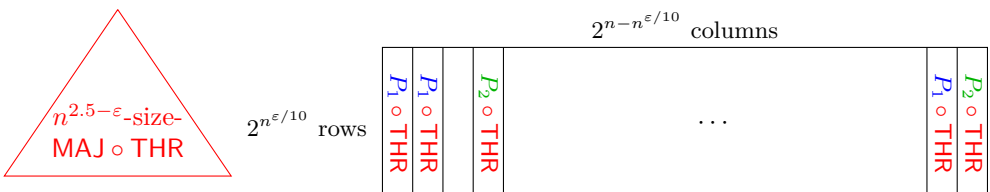

- **Our observation:** There are only $n^{10 \log n}$ many different polynomials!
- For each $i \in \left[n^{10 \log n}\right]$, compute $P_i \circ$ THR $-$ CAPP on all columns;
- Then for each column $z \in \{0,1\}^{n-n^{\varepsilon/10}}$, look up the corresponding $P_i \circ$ THR $-$ CAPP.

[Alman-Williams' 15]'s Proof

- SUM with 1-hot encoding of output (1hotSUM):
  - Input: $(x_1, \cdots, x_m) \in \{0,1\}^m$;
  - Output: $(y_0, y_1, \cdots, y_m) \in \{0,1\}^{m+1}$:
    If $\sum_{i \in [m]} x_i = s$, then only the s-th bit is 1, the others are 0.

Theorem

1hotSUM *of m bits can be approximated by degree-$\widetilde{O}(\sqrt{m})$ polynomials over $\mathbb{F}_2$.*

- Pointwise-Approximation: there is a distribution $\mathcal{D}$ over $(m+1)$-tuple of degree-$\widetilde{O}(\sqrt{m})$ polynomials such that:

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$$P_k(x) := \sum_{s=k/2-\widetilde{O}(\sqrt{m})}^{k/2+\widetilde{O}(\sqrt{m})} Q_s(x_A) \cdot \underbrace{\left(\begin{cases} 1 & \text{if } \sum_{i \in [m]} x_i = k \\ 0 & \text{if } \sum_{i \in [m]} x_i \in [2s \pm \widetilde{O}(\sqrt{m})] \setminus \{k\} \\ \text{any} & \text{otherwise} \end{cases} \right)}_{\text{interpolation polynomial}}.$$

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  - In total: #rounds $= \log m$, degree: $\widetilde{O}(\sqrt{m}) \cdot \log m = \widetilde{O}(\sqrt{m})$.

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- **a-s of each induction step encodes everything we need to know about X !**
  Encoding length: $O\left((\log n)^2\right)$; #different polynomials: $n^{O(\log n)}$.

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- **Open Problems:**
  - Push forward circuit lower bounds?
  - Other applications of list approximation polynomials?

Thank You

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